

Loewner equation for Laplacian growth: A Schwarz-Christoffel-transformation approach

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Abstract. The problem of Laplacian growth is considered within the Loewner-equation framework. A new method of deriving the Loewner equation for a large class of growth problems in the half-plane is presented. The method is based on the Schwarz-Christoffel transformation between the so-called ‘mathematical planes’ at two infinitesimally separated times. Our method not only reproduces the correct Loewner evolution for the case of slit-like fingers but also can be extended to treat more general growth problems. In particular, the Loewner equation for the case of a bubble growing into the half-plane is presented.

1. Introduction

The Loewner equation [1] is an important result in the theory of univalent functions [2] that has found important applications in nonlinear dynamics, statistical physics, and conformal field theory [3, 4]. In its most basic formulation, the Loewner equation is a first-order differential equation for the conformal mapping $g_t(z)$ from a given ‘physical domain,’ consisting of a complex region \mathbb{P} minus a curve Γ_t emanating from its boundary, onto a ‘mathematical domain’ represented by \mathbb{P} itself. Usually, \mathbb{P} is either the upper half-plane or the exterior of the unit circle, but recently the Loewner equation for the channel geometry was also considered [5]. The Loewner equation depends on a driving function, here called $a(t)$, that is the image of the growing tip under the mapping $g_t(z)$. An important development on the theory of the Loewner equation was the discovery by Schramm [6] that when the driving function $a(t)$ is a Brownian motion the resulting Loewner evolution describes the scaling limit of certain statistical mechanics models. This result spurred great interest in the so-called stochastic Loewner equation [4].

Recently, the deterministic Loewner equation was also used to study the problem of Laplacian fingered growth in both the half-plane and radial geometries [7, 8] as well as in the channel geometry [5]. In this case, the driving function $a(t)$ has to follow a specific time evolution in order to ensure that the finger tip grows along the gradient lines of the corresponding Laplacian field. The idea of using iterated conformal maps to generate aggregates was first deployed by Hastings and Levitov [9] in the context of stochastic growth models, such as diffusion limited aggregation; see, e.g., [10] for further developments along those lines. A deterministic version of the Hastings-Levitov model that is closely related to the Loewner-equation approach—albeit not using explicitly such a formalism—was studied in [11].

In this paper we consider the problem of Laplacian growth within the context of the Loewner evolution, and present a new method of deriving the corresponding Loewner equation for a broad

class of growth models in the half-plane. Our method is based on the Schwarz-Christoffel (SC) transformation between the mathematical planes g_t and $g_{t+\tau}$, where τ is an infinitesimal time interval. More specifically, the method consists of expanding the integrand of the SC formula in powers of the appropriate infinitesimal quantity (related to τ) and then performing the integrals up to the leading-order term. Our method correctly yields the Loewner evolution for the case of slit-like fingers studied before [5, 8]. More importantly, the method is able to handle more general growth problems, so long as the growth rule can be specified (in the mathematical plane) in terms of a polygonal curve, in which case the Schwarz-Christoffel transformation can be used. An example is given for the case of a bubble growing from the real axis into the upper half-plane.

2. Loewner Equation for Slit-like Fingers

2.1. The Case of a Single Finger

In order to set the stage for the remainder of the paper, we wish to begin our discussion by considering the simplest Loewner evolution, namely, that in which a curve starts from the real axis at $t = 0$ and then grows into the upper half- z -plane \mathbb{H} , where

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}.$$

The curve at time t is denoted by Γ_t and its growing tip is labeled by $\gamma(t)$. Now let $g_t(z)$ be the conformal mapping that maps the ‘physical domain,’ corresponding to the upper half- z -plane minus the curve Γ_t , onto the upper half-plane of an auxiliary complex w -plane, called the ‘mathematical plane,’ i.e., we have $w = g_t(z)$, where

$$g_t : \mathbb{H} \setminus \Gamma_t \rightarrow \mathbb{H}, \quad (1)$$

with the curve tip $\gamma(t)$ being mapped to a point $a(t)$ on the real axis in the w -plane; see Fig. 1. Furthermore, we consider the growth process to be such that the accrued portion of the curve from t to $t + \tau$, where τ is an infinitesimal time interval, is mapped under $g_t(z)$ to a vertical slit in the mathematical w -plane; see Fig. 1. The mapping function $g_t(z)$ must also satisfy the initial condition

$$g_0(z) = z, \quad (2)$$

since we start with an empty upper half-plane. We also impose the so-called hydrodynamic normalization condition at infinity:

$$g_t(z) = z + O\left(\frac{1}{|z|}\right), \quad z \rightarrow \infty. \quad (3)$$

These conditions specify uniquely the mapping function $g_t(z)$.

From a more physical viewpoint, the problem formulated above belong to the class of Laplacian growth models where an interface evolves between two phases driven by a scalar field $\phi(x, y; t)$, representing, for example, temperature, pressure, or concentration, depending on the physical problem at hand. In one phase, initially occupying the entire upper half-plane, the scalar field ϕ satisfies the Laplace equation

$$\nabla^2 \phi = 0, \quad (4)$$

whereas in the other phase one considers $\phi = \text{const.}$, say $\phi = 0$, with the curve Γ_t representing a finger-like advancing interface between the two phases. (Here the finger is assumed to be infinitesimally thin.) The complex potential for the problem can then be defined as $w(z, t) = \psi(x, y; t) + i\phi(x, y; t)$, where ψ is the function harmonically conjugated to ϕ . On the boundary of the physical domain, consisting here of the real axis together with the curve Γ_t ,

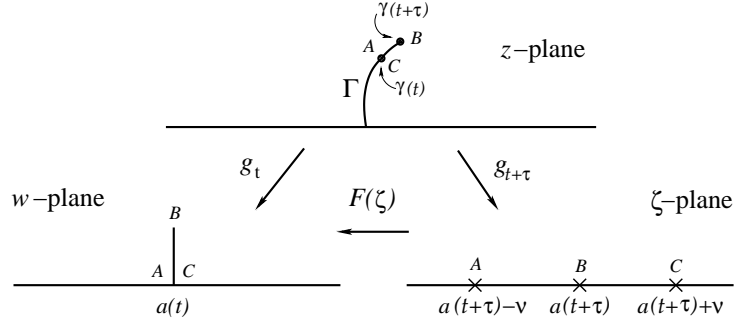


Figure 1. The physical z -plane and the mathematical w - and ζ -planes at times t and $t + \tau$, respectively, for a single finger in the upper half-plane. The mapping g_t maps the curve Γ (at time t) onto a segment of the real axis on the w -plane, whereas the accrued portion of the curve (during time interval τ) is mapped to a vertical slit. The mapping g_t is obtained as the composition of $g_{t+\tau}$ and the slit mapping F ; see text.

we impose the condition $\phi = 0$, whereas at infinity we assume a uniform gradient field, $\vec{\nabla}\phi \approx \hat{y}$, or alternatively,

$$w(z, t) \approx z, \quad z \rightarrow \infty. \quad (5)$$

From this point of view, the mapping function $g_t(z)$ introduced above corresponds precisely to the complex potential $w(z, t)$ of the problem. In particular, the fact that in the w -plane the curve grows along a vertical line implies that the finger tip grows along gradient lines in the z -plane. To specify completely a given physical model, one has also to prescribe the interface velocity, which is usually taken to be proportional to some power η of the gradient field: $v \sim |\vec{\nabla}\phi|^\eta$. For most of the problems considered here the specific velocity model is not relevant, in the sense that the finger shapes will be independent of the exponent η , which only affects the time scale of the problem. (However, there are situations, such as the case of competing asymmetrical fingers [5], where different η 's may yield different patterns.)

For convenience of notation, we shall represent the mathematical plane at time $t + \tau$ as the complex ζ -plane and so we write $\zeta = g_{t+\tau}(z)$. Now consider the mapping $w = F(\zeta)$, from the upper half- ζ -plane onto the mathematical domain in the w -plane; see Fig. 1. The mapping function $g_{t+\tau}(z)$ can then be given in terms of $g_t(z)$ as

$$g_{t+\tau} = F^{-1} \circ g_t, \quad (6)$$

where F^{-1} is the inverse of $F(\zeta)$. The above relation governs the time evolution of the function $g_t(z)$ and naturally leads to the Loewner equation. A standard way of showing this is to construct the slit mapping $F(\zeta)$ explicitly, substitute its inverse in (6), and then take the limit $\tau \rightarrow 0$. One then finds the Loewner equation

$$\dot{g}_t(z) = \frac{d(t)}{g_t(z) - a(t)}, \quad (7)$$

where $d(t)$ is the so-called growth factor which is related to the tip velocity. One can show [5] that $d(t) = |f_t''(a(t))|^{-\eta/2-1}$, where $f_t(z)$ is the inverse of $g_t(z)$. Here, however, the specific form of $d(t)$ is not relevant since we can always rescale the time coordinate in (7) so as to set $d(t) = 1$. From symmetry, one also gets

$$\dot{a}(t) = 0, \quad (8)$$

so that $a(t) = a_0 = \text{const.}$, which implies that the tip $\gamma(t)$ simply traces out a vertical line in the z -plane.

In more general situations, such as the case of multiple fingers, the function $F(\zeta)$ can no longer be obtained in closed form and so one has to resort to alternative approaches to derive the Loewner evolution. Previous methods [5, 8] consider a series of compositions of the basic one-slit mapping. Here, however, we will apply a more direct method based on the Schwarz-Christoffel transformation to obtain the Loewner equation for slit-like fingers in the half-plane. In the next section, our method will be extended to treat the case of a growing bubble in the upper half-plane. To illustrate how the method works, let us first use it to derive the Loewner equation (7) for a single finger.

We begin by inverting (6) so as to write

$$g_t = F \circ g_{t+\tau}. \quad (9)$$

The mapping $F(\zeta)$ from the upper half- ζ -plane onto the upper half- w -plane with a vertical slit (see Fig. 1) is easily found by a direct application of the Schwarz-Christoffel formula [12]. One then finds

$$g_t = F(g_{t+\tau}) = \int_{\zeta_0}^{g_{t+\tau}} \frac{\zeta - a(t+\tau)}{\sqrt{(\zeta - a(t+\tau))^2 - \nu^2}} d\zeta + a(t), \quad (10)$$

where $\zeta_0 = a(t+\tau) + \nu$. Note from Fig. 1 that the parameter ν is related to the (infinitesimal) height of the slit in the w -plane. The above integral can be performed exactly, as already mentioned, but here we take an alternative approach, namely, we first expand the integrand in (10) in powers of ν and then compute the relevant integrals afterwards. To do that we first rewrite (10) in the form

$$g_t = \int_{\zeta_0}^{g_{t+\tau}} \frac{d\zeta}{\sqrt{1 - \nu^2[\zeta - a(t+\tau)]^{-2}}} + a(t). \quad (11)$$

After expanding the integrand in powers of ν^2 and performing the relevant integrals, one obtains that, up to order ν^2 , equation (11) becomes

$$g_{t+\tau} - g_t = \frac{\nu^2/2}{g_{t+\tau} - a(t+\tau)} + a(t+\tau) - a(t). \quad (12)$$

Now expanding this equation up to the first order in τ , dividing by τ , and then taking $\tau \rightarrow 0$, one gets

$$\dot{g}_t = \frac{d(t)}{g_t(z) - a(t)} + \dot{a}, \quad (13)$$

where

$$d(t) = \lim_{\tau \rightarrow 0} \frac{\nu^2}{2\tau}. \quad (14)$$

Using the boundary condition $\lim_{g_t \rightarrow \infty} \dot{g}_t = 0$, which follows from (3), yields precisely the Loewner equation (7) together with the condition (8).

2.2. The Multifinger Case

Here we consider the case of multiples fingers Γ_i , $i = 1, \dots, n$, growing from the real axis into the upper half- z -plane, as shown in Fig. 2. As before, the map $g_t(z)$ maps the physical domain in the z -plane onto the upper half- w -plane and the tips are required to grow along gradient lines, so that the accrued portions of the curves Γ_i during an infinitesimal time τ are mapped under $g_t(z)$ to vertical slits emanating from the real axis; see Fig 2.

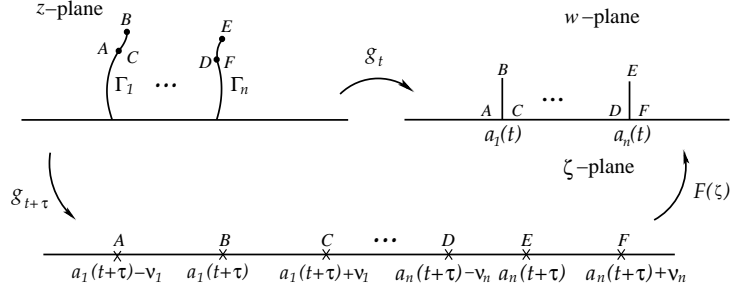


Figure 2. The physical and mathematical planes for the case of multiple fingers in the upper half-plane.

The mapping $w = F(\zeta)$, from the upper half- ζ -plane to the upper half- w -plane with n vertical slits, can again be easily obtained from the Schwarz-Christoffel transformation [12]:

$$g_t = F(g_{t+\tau}) = \int_{\zeta_0}^{g_{t+\tau}} \prod_{i=1}^n \frac{\zeta - a_i(t+\tau)}{\sqrt{(\zeta - a_i(t+\tau))^2 - \nu_i^2}} d\zeta + a_j(t), \quad (15)$$

where $\zeta_0 = a_j(t+\tau) + \nu_j$ for a given j . [We remark parenthetically that in writing (15) we have assumed, for simplicity, that the slits in the w -plane are mapped under $F^{-1}(w)$ onto symmetrical segments on the real ζ -axis; see Fig 2. Rigorously speaking, this is valid only in the limit that the slit heights become vanishingly small, that is, when $\tau \rightarrow 0$, which is the relevant limit for us here.]

The integral in (15) cannot be performed exactly for arbitrary n , hence in order to obtain the Loewner equation for this case we first need to expand the integrand in powers of the infinitesimal parameters ν_i and then proceed with the integration. Note, however, that each term in (15) is of the same form as that appearing in (10) for the case of a single finger. We can thus build upon our experience with that case to treat the present situation. In particular, we notice that the mixed terms involving different ν_i 's in the expansion of the integrand in (15) are all of orders higher than ν^2 and hence need not be considered, for they do not contribute to the final result in the limit $\tau \rightarrow 0$. Thus, to the extent that the mixed terms can be neglected, we can rewrite (15) as

$$g_t \approx \sum_{i=1}^n \int_{\zeta_0}^{g_{t+\tau}} \frac{d\zeta}{\sqrt{1 - \nu_i^2 [\zeta - a_i(t+\tau)]^2}} + a_j(t). \quad (16)$$

Now repeating the exactly same procedure used for the single finger case, see Eqs. (11)-(13), one readily obtains

$$\dot{g}_t = \sum_{i=1}^n \frac{d_i(t)}{g_t - a_i(t)} - \sum_{\substack{i=1 \\ i \neq j}}^n \frac{d_i(t)}{a_j(t) - a_i(t)} + \dot{a}_j(t), \quad (17)$$

where

$$d_i(t) = \frac{\nu_i^2}{2\tau}. \quad (18)$$

After using the condition $\lim_{g_t \rightarrow \infty} \dot{g}_t = 0$ in (17), we get the Loewner equation for multiple curves

$$\dot{g}_t = \sum_{i=1}^n \frac{d_i(t)}{g_t - a_i(t)}, \quad (19)$$

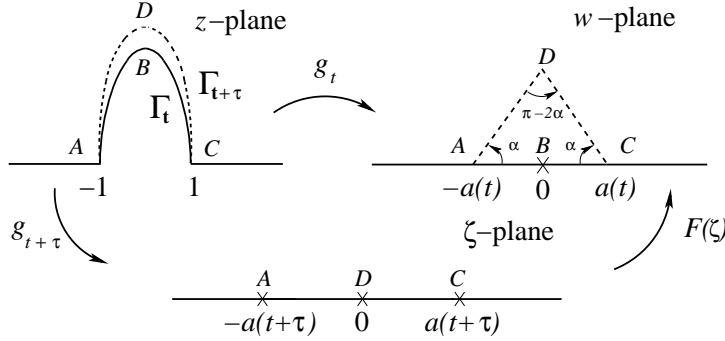


Figure 3. The physical and mathematical planes for the case of a growing bubble in the upper half-plane. Here the new interface at time $t + \tau$ is mapped under g_t to a tent-like shape; see text for details.

with the time evolution of the points $a_i(t)$ being given by the following system of ordinary differential equations

$$\dot{a}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^n \frac{d_j(t)}{a_i(t) - a_j(t)}. \quad (20)$$

If the growth factors $d_i(t)$ are all the same, we can again rescale the time variable so as to set $d_i = 1$. In particular, in the case of two symmetrical fingers (i.e., $d_1 = d_2 = 1$), equation (19) can be integrated exactly to yield the mapping function $g_t(z)$, from which the finger shapes can be computed analytically [5]. A related exact solution for two fingers was obtained in [13]. An alternative derivation of (19) was given elsewhere [8] using a composition of n single-slit mappings. Our method, however, is somewhat more direct in the sense that it considers a single mapping with n slits as shown in Fig 2. In the next section we will extend our method to include the case of a growing bubble in the half-plane.

3. Loewner Equation for a Bubble in the Half-Plane

Here we consider the problem of an interface starting initially from a segment, say, $[-1, 1]$, along the real axis and then growing into the upper half- z -plane, as indicated in Fig. 3. As before, we denote the interface at time t by Γ_t and consider the mapping $g_t(z)$ from the physical domain in the z -plane to the mathematical w -plane:

$$g_t : \mathbb{H} \setminus \Gamma_t \rightarrow \mathbb{H}.$$

The mapping function $g_t(z)$ must also satisfy the hydrodynamic normalization condition (3). The growth dynamics is specified by requiring that each point on the interface grow along gradient lines, but with a growth rate that is maximum at the center (i.e., at the bubble tip) and zero at both ‘contact points,’ $z = \pm 1$, in such a way that the interface at time $t + \tau$ is mapped under $g_t(z)$ to a tent-like shape, as shown in Fig. 3. The height of this tent, or alternatively, the angle α formed with the real axis, is linearly related to τ in the limit $\tau \rightarrow 0$, as we will see below. Here we consider only the case of symmetric bubbles, where the contact points, $z = \pm 1$, are mapped by $g_t(z)$ to the points $w = \pm a(t)$, respectively, and the tip (point B in Fig. 3) is mapped to the origin.

Since the domain in the w -plane has a polygonal shape, the mapping $w = F(\zeta)$ can once again be obtained from the Schwarz-Christoffel transformation. In this case we have

$$g_t = F(g_{t+\tau}) = \int_{a(t+\tau)}^{g_{t+\tau}} \frac{\zeta^{2\alpha/\pi}}{[\zeta^2 - a^2(t+\tau)]^{\alpha/\pi}} d\zeta + a(t), \quad (21)$$

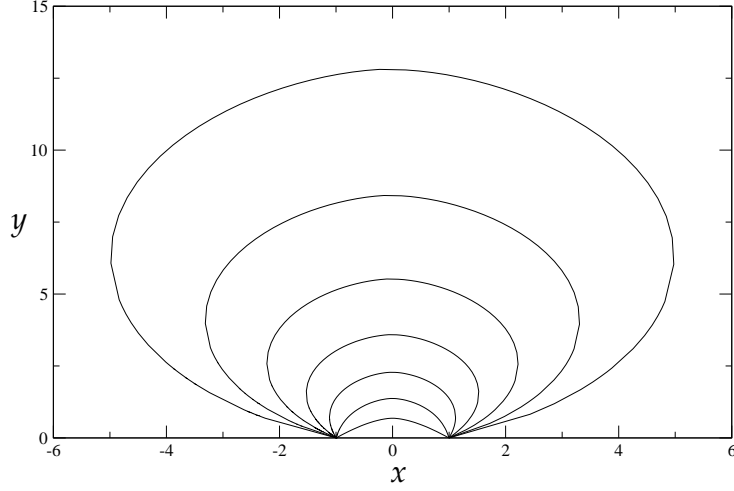


Figure 4. Loewner evolution for a bubble growing in the upper half-plane for $d(t) = 1$. The bubble starts at $t = 0$ from the interval $[-1, 1]$ on the real axis. The different curves represent the interface at time intervals of $\Delta t = 0.3$.

which up to the first order in the parameter α becomes

$$g_t \approx \int_{a(t+\tau)}^{g_{t+\tau}} \left\{ 1 + \frac{\alpha}{\pi} [2 \ln \zeta - \ln(\zeta^2 - a^2(t + \tau))] \right\} d\zeta + a(t), \quad (22)$$

After performing the integral above and taking the limit $\tau \rightarrow 0$, one readily obtains the following generalized Loewner equation:

$$\dot{g}_t(z) = d(t) \{ [g_t + a(t)] \ln[g_t + a(t)] + [g_t - a(t)] \ln[g_t - a(t)] - 2g_t \ln g_t \}, \quad (23)$$

together with the governing equation for the logarithmic singularity $a(t)$

$$\dot{a}(t) = (\ln 4) d(t) a(t), \quad (24)$$

where the growth factor $d(t)$ is now given by

$$d(t) = \lim_{\tau \rightarrow 0} \frac{\alpha}{\pi \tau}. \quad (25)$$

Equation (24) can be solved exactly to yield $a(t) = a_0 4^{\int_0^t d(t') dt'}$. As explained before, we can set $d(t) = 1$ by rescaling the time coordinate, in which case we have $a(t) = a_0 4^t$. Recall also that the Loewner equation (23) must be supplemented with the initial condition $g_0(z) = z$.

In Fig. 4 we show numerical solutions of (23) with $a_0 = 1$ for various final times t , starting from $t = 0.3$ up to $t = 2.1$ with a time separation of $\Delta t = 0.3$ between successive curves. To generate the curves shown in Fig. 4 we used the numerical scheme described in [13]. More specifically, we start with a ‘terminal condition’ $g_t = w$, for $w \in [-a(t), a(t)]$, and integrate the Loewner equation (23) backwards in time, using a Runge-Kutta method of second order, to get the corresponding point g_0 on the interface. (See also [14] for a recent review on numerical integration of the Loewner equation.) From Fig. 4 one sees that the bubble initially grows somewhat slowly and then rapidly expands and tends to occupy the whole plane for large times. In fact, one can show that (for $d = 1$) the tip velocity grows exponentially with time as $t \rightarrow \infty$. It is possible to modify the growth factor $d(t)$ so to have the tip velocity related to the gradient of the field $\phi(x, y)$, as discussed in Sec. 2, but this does not change the interface shapes and only alters the time scale of the bubble evolution.

4. Conclusions

We have presented a novel method to derive the Loewner equation for Laplacian growth problems. The method is based on the Schwarz-Christoffel (SC) transformation and consists of expanding the integrand of the SC formula in the appropriate infinitesimal parameter, performing the relevant integrals, and then taking the limit in which the infinitesimal parameter goes to zero. Our method is able to reproduce the Loewner evolution for the problem of slit-like fingers in both the half-plane (Sec. 2) and the channel geometry (not shown here). Furthermore, the method can be extended to treat more complicated growth problems, so long as the growth dynamics in the complex-potential plane can be specified in terms of a polygonal boundary, in which case the Schwarz-Christoffel transformation can be used. We note however that the requirement that the growth rule be formulated in terms of a polygonal curve is not as restrictive as it seems, for any simple curve can in principle be approximated by a piecewise linear function. [Such more general growth models are currently under investigation.] In particular, we obtained the Loewner equation for a novel situation in which a bubble grows from a segment of the real line into the upper half-plane. Although in this case we refer to the evolving interface as a growing ‘bubble,’ in contrast to the slit-like ‘fingers’ of Sec. 2, this terminology should not be taken literally. Depending on the physical problem at hand, such growing interface may represent, say, an expanding front in combustion experiments or in electrochemical deposition [5]. Of course, further work is necessary to relate more directly the growth models discussed here to experiments.

Acknowledgments

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